The Mathematical Mystery of Odd Perfect Numbers

An Exploration of the Odd Perfect Number Problem, Multiply Perfect Numbers, and Related Topics in Number Theory

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Introduction and Background

A perfect number is a number that is equal to the sum of its proper divisors. For example, 6 is a perfect number because its proper divisors are 1, 2, and 3 and $6 = 1 + 2 + 3$. A non-example of a perfect number is 8, whose proper factors sum to $1 + 2 + 4 = 7 \neq 8$. The next smallest perfect number after 6 is $28 = 1 + 2 + 4 + 7 + 14$ and after 28 we have 496. There are 52 known perfect numbers. The largest one was discovered in October 2024 and has more than 82 million digits ([10]).

All 52 known perfect numbers are even. Given this observation, it is natural to ask, do odd perfect numbers exist? Indeed, many people have asked this question before. Number theorists have been wondering whether or not odd perfect numbers exist for more than 2,000 years and yet we remain unable to rigorously answer the problem. Our understanding of odd perfect numbers provides a stark contrast to how well-studied and formulated even perfect numbers are. Every few years increased computational power partnered with our understanding of even perfects allows us to find a new even perfect number. The more we search for odd perfect numbers, the more elusive they seem.

We will be talking about the odd perfect number problem and a smorgasbord of related problems, such as multiply perfect numbers, super perfect numbers, and primitive abundant numbers in our literature review. Before we do so, we will formally define all the terms and functions we need in the Theoretical Frameworks section, which can be used as a reference for these terms as we proceed through the rest of the paper. Our discussion will culminate in my original work with non-divisors of odd perfect numbers and determining the possible forms of multiply perfect numbers.

Theoretical Frameworks

Before we can discuss important theorems, conjectures, and new findings surrounding odd multiply perfect numbers we need to establish the notation that we will be using and define important terms using formal mathematics. The Theoretical Frameworks section serves that purpose and acts as a place of reference for definitions as we get into the Literature Review and Original Work sections.

First, we will define the σ-function and the abundancy index as these two functions simplify working with multiply perfect numbers and have nice properties.

Definition 1 (The σ-function) *Define the function σ such that $\sigma(n)$ is equal to the sum of all of the divisors of n, including 1 and n itself, where $n \in \mathbb{N}$.*

As an example of the sigma function, $\sigma(8) = 1 + 2 + 4 + 8 = 15$, because the divisors of 8 are 1, 2, 4, and 8. As an additional example, $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$.

Definition 2 (The Abundancy Index) *Define the function I such that $I(n) = \frac{\sigma n}{n}$, where $n \in \mathbb{N}$. Note that $I(n)$ is called 'the abundancy index of n.'*

As an example of the abundancy index in action, $I(8) = \frac{\sigma(8)}{8} = \frac{15}{8} = 1.875$.

We now define gcd aka the greatest common divisor function.

Definition 3 (Greatest Common Divisor) *Let $a, b \in \mathbb{N}$. Then the greatest common divisor of a and b, written as* $\gcd(a, b)$*, is the largest integer that divides both a and b. Furthermore, a and b are called relatively prime if and only if* $\gcd(a, b) = 1$.

Observe that $\gcd(12, 16) = 4$ since the largest integer that divides 12 and 16 is 4. Also, $\gcd(9, 20) = 1$ since 9 and 20 have no common divisors aside

from 1, meaning that 9 and 20 are relatively prime. Let us also define what it means for a number to be prime.

Definition 4 (Prime) *A prime number $p \in \mathbb{N}$ has exactly two distinct divisors, which are 1 and p.*

Note that 1 is not a prime number because 1 has exactly one distinct divisor (which is itself). The smallest prime number is 2 and there are infinitely many prime numbers as stated in Theorem 1. Theorem 1 was first proved by Euclid and is well known.

Theorem 1 *There are infinitely many prime numbers.*

Lemmas 1, 2, and 3 give some useful characteristics of σ. Lemma 1 follows directly from the definition of the σ-function and the behavior of primes raised to a power.

Lemma 1 (Euler and the σ-function Part 1) *If p is a prime number and $k \in \mathbb{N}$, then $\sigma(p^k) = 1 + p + p^2 + \cdots + p^k$.*

Lemma 2 makes use of the greatest common divisor function which we defined earlier.

Lemma 2 (Euler and the σ-function Part 2) *If $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$, then $\sigma(a \cdot b) = \sigma(a) \cdot \sigma(b)$. This means that σ has the multiplicative property.*

Lemma 3 follows directly from the formula for geometric series, which can be found in any elementary number theory textbook such as [3].

Lemma 3 (Extension of Lemma 1) *If p is a prime number and $k \in \mathbb{N}$, then $\sigma(p^k) = \frac{p^{k+1}-1}{p-1}$.*

We are going to be heavily discussing multiply perfect numbers, so let's define them in Definition 5.

Definition 5 (Multiply Perfect Numbers) *Let $M \in \mathbb{N}$. Then M is multiply perfect if and only if $I(M) \in \mathbb{Z}$. Additionally, M is called n-perfect for $n \in \mathbb{N}$ if and only if $\sigma(M) = n \cdot M \Leftrightarrow I(M) = n$.*

Note that 2-perfect numbers are commonly referred to as simply 'perfect' and that 3-perfect numbers are commonly called 'triperfect.'

We will make reference to abundant and deficient numbers as defined in Definition 6. The definition of the abundancy index is closely tied to Definition 6.

Definition 6 (Abundant and Deficient) *A number n is called abundant if $I(n) > 2$ and is called deficient if $I(n) < 2$.*

An extension of abundant numbers is primitive abundant numbers, defined in Definition 7.

Definition 7 (Primitive Abundant Numbers) *A number n is called a primitive abundant number if it is non-deficient and if it is not a multiple of any smaller non-deficient number.*

The set of perfect numbers is a subset of the set of primitive abundant numbers.

Lemma 4 is logically equivalent to a theorem that appears in [7] and will be important to us later.

Lemma 4 *For all $m, n \in \mathbb{N}$ with $m \neq 1$, $I(mn) > I(n)$.*

Now we will formally define divides, double bar divides (which is an extension of divides), and modular arithmetic.

Definition 8 (Divides) *Given $a, b \in \mathbb{Z}$, we write $a \mid b$ or say 'a divides b' if and only if there exists $k \in \mathbb{Z}$ such that $a \cdot k = b$. Otherwise, we write $a \nmid b$.*

As an example of divides, we say $4 \mid 36$ since $9 \in \mathbb{Z}$ and $4 \cdot 9 = 36$ however $4 \nmid 26$ since there is no $k \in \mathbb{Z}$ such that $4 \cdot k = 26$.

Definition 9 (Double Bar Divides) *Given $a, b \in \mathbb{Z}$, we write $a \parallel b$ if and only if $a \mid b$ and* $\gcd(a, \frac{b}{a}) = 1$.

As an example of double bar divides, $10 \parallel 30$ since $10 \mid 30$ and $\frac{30}{10} = 3$ is relatively prime to 10.

Definition 10 (Mod) *We write $n \equiv r \mod q$ and say 'n is congruent to r mod q' for $n, q, r \in \mathbb{Z}$ if and only if $q \mid (n - r)$ or, equivalently, there exists $k \in \mathbb{Z}$ such that $a = q \cdot k + r$.*

Modular arithmetic is far simpler than it sounds when defined formally. As an example $5 \equiv 1 \mod 4$ since $4 \mid (5 - 1)$ but $7 \not\equiv 2 \mod 4$ since $4 \nmid (7 - 2)$.

Another set of numbers that will come up in our conversation is the triangular numbers, which are defined in Definition 11.

Definition 11 (Triangular numbers) *A triangular number is equal to* $1+2+3+...+n$ *for some n in the natural numbers.*

The triangular numbers are 1, 1+2=3, 1+2+3=6, 1+2+3+4=10, 1+2+3+4+5=15, 1+2+3+4+5+6=21, etc.

While lim sup does not play a huge role, it will come up later in our discussion of the σ function. Note that a sequence is simply an ordered, infinite set of numbers. For example, the triangular numbers (1,3,6,10,15,...) is a sequence.

Definition 12 *Given a sequence $(a_n) = (a_1, a_2, a_3, ...)$ that is bounded above,* $\limsup a_n$ *is the smallest upper bound of (a_n).*

Full understanding of lim sup is outside of the scope of our analysis.

There will be some more definitions and lemmas that come up later, but we now have the necessary items in our toolkit to proceed into our literature review.

Literature Review

Most number theorists who have studied the odd perfect number problem generally come to the conjecture that odd perfect numbers cannot exist. As John Voight, professor of mathematics at Dartmouth College, said "proving that something exists is easy if you can find just one example, but proving that something does not exist can be really hard" ([18]). James Joseph Sylvester, who in 1888 proved that an odd perfect number cannot be divisible by 105 and that an odd perfect number must have at least three distinct prime factors, reflected "the existence of [an odd perfect number]–its escape, so to say, from the complex web of conditions which hem it in on all sides–would be little short of a miracle" ([18]).

The majority of theorems regarding odd perfect numbers follow the structure of 'If an odd perfect number exists, then [conclusion].' To prove that an odd perfect number does not exist it would suffice to prove two such theorems with contradictory conclusions hold. Most of the theorems that have been proven pertaining to odd perfect numbers were written with the hope of finding a contradiction. In the 2,000 years that odd perfect numbers have been recognized and studied, mathematicians have established "an extraordinary list of restrictions," but no contradiction has yet been reached ([18]).

If an odd perfect number exists then it is larger than 10^{1500} ([20]). For comparison, there are only about 10^{82} atoms in the observable universe. An odd perfect number would have to be an incredibly large number. The lower bound for odd perfect numbers is constantly being improved as increased computational power can be applied to the problem. A paper published in 1949 gives that the lower bound for odd perfect numbers is 2,000,000,000 or $2 \cdot 10^9$ showing just how much the bound has been improved in the last 75

years ([1]). Further, the largest prime power in an OPN's prime decomposition is greater than 10^{62} and it must have at least 101 prime factors ([20]).

Spoof Perfect Numbers

The set of odd perfect numbers is a subset of the set of odd 'spoof' perfect numbers. A spoof perfect number is a number that would be perfect if you ignored one shortcoming. For example, the first spoof perfect number was discovered by Descartes in 1638 and is $198,585,576,189 = 3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 22,021^1$ ([18]). Descartes' number would be a perfect number under the multiplicative property of the sigma function if only 22,021 were prime. The second spoof odd perfect number was discovered 361 years later in 1999 by John Voight and is $-22,017,975,903 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2 \cdot (-127)^1$, which would be perfect if we pretend that -127 isn't negative.

Voight, along with a team of other math professors and students at BYU, completed a study of spoof perfect numbers in 2019. The team used 20 parallel processors over the course of three years to find all possible spoof perfect numbers with factorizations of six or fewer bases ([18]). They found a total of 21 spoofs (including Descartes' and Voight's original two examples) and an additional two spoofs with factorizations of seven bases ([18]). The BYU team also found that there are infinitely many spoofs, but there are only a finite number of them for any fixed number of bases. In a paper titled "Spoof Odd Perfect Numbers" published in 2014 ([8]), the author, Samuel J. Dittmer, thanked the BYU Mathematics Department for "support of this research," showing the significant role that this team has played in the research surronding spoof perfect numbers.

Dittmer defines spoof perfect numbers slightly more specifically than we have. Definition 13 is logically equivalent to Dittmer's definition of spoof

perfect numbers.

Definition 13 (Dittmer's definition of spoof perfect numbers) *A spoof perfect number S can be written as $S = p_1^{a_1} \cdot p_2^{a_2} \cdot ... \cdot p_k^{a_k}$ where $k \in \mathbb{N}$ and*

$$2S = \frac{p_1^{a_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{a_2+1} - 1}{p_2 - 1} \cdot ... \cdot \frac{p_k^{a_k+1} - 1}{p_k - 1}.$$

We refer to the p_i as 'quasi-primes.'

Notice that Dittmer's definition compared to the definition of a perfect number only relaxes the fact that every base in the factorization of the number must be a distinct prime. Dittmer calls the bases 'quasi-primes' because they may or may not be primes but, regardless, they are treated like primes under the sigma function in order to produce a spoof perfect number (note that $\sigma(p^n) = \frac{p^{n+1}-1}{p-1}$ when p is prime by Lemma 3). Unlike Voight, Dittmer does not include negative spoofs in his definition.

Using his definition of spoof perfect numbers, Dittmer proved Theorem 2 ([8]).

Theorem 2 *The only spoof odd perfect number (defined under Definition 13) with less than seven quasi-prime factors is Descartes' example,* $3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 22021^1$.

Dittmer also found an algorithm for generating odd spoofs and he showed that there are infinitely many even spoofs. In [8], he gives three infinite families of even spoof perfect numbers which we give as Theorem 3.

Theorem 3 *For all $n > 1$ and $\alpha \geq 1$, the following are even spoof perfect*

numbers.

$$2^{n-1}(2^n - 1)^1 \tag{1}$$

$$n^1(n+1)^1 ... (2n-1)^1 \tag{2}$$

$$n^\alpha \cdot \left(\frac{n^{\alpha+1}-1}{n-1}\right)^1 \cdot \left(\frac{n^{\alpha+1}-1}{n-1}+1\right)^1 ... (2 \cdot n^\alpha - 1)^1 \tag{3}$$

Notice that, with reference to Theorem 16, that the set of all even perfect numbers is a subset of the set of numbers of the form given by equation .

Since the set of odd perfect numbers is a subset of the set of odd spoofs, any property that is proven to hold true for odd spoofs must also hold for odd perfects. In this way Voight, the BYU team, and Dittmer successfully tightened the net on the odd perfect number problem.

Primitive Abundant Numbers

The set of perfect numbers is a subset of the set of primitive abundant numbers, so these two sets share quite a few characteristics. Per Definition 7, a primitive abundant number is a non-deficient number that is not divisible by any other non-deficient numbers. The smallest odd primitive abundant number is 945 and the smallest even primitive abundant number is 20. A list of other odd primitive abundant numbers is given in [6]. By Theorem 4, the abundancy of any multiple of a perfect number, similar to the rest of the primitive abundant numbers, will be greater than two, or abundant.

The main result of [6] is given in Theorem 4.

Theorem 4 *There is only a finite number of primitive non-deficient odd numbers having any given number of distinct prime factors.*

Theorem 4 suggests, as Leonard Eugene Dickson points out in a corollary in [6], there cannot be infinitely many "odd perfect numbers with any given number of distinct prime factors." Dickson emphasizes in the footnotes that

Theorem 4 only applies to odd numbers since "if p is prime $2^m p$ is non-deficient if and only if $2^{m+1} > p+1$," which shows that there can be infinitely many even primitive abundant numbers with a given number of distinct prime factors [6].

In [23] Paul Pollack expands upon a theorem of L. E. Dickson in [6]. We gave Dickson's theorem as Theorem 4 and we give Pollack's generalization of Dickson's theorem as Theorem 5.

Theorem 5 *For each positive integer k, the number of odd perfect numbers N where the number of distinct prime factors of N is less than or equal to k is bounded by* 4^{k^2}.

Multiply Perfect Numbers

Yet another superset of perfect numbers is multiply perfect numbers. We defined multiply perfect numbers in Definition 5. The smallest ten multiply perfect numbers are 1, 6, 28, 120, 496, 672, 8128, 30240, 32760, and 523776. Of this list, 1 is 1-perfect, 6, 28, 496, and 8128 are 2-perfect, 120, 672, and 523776 are 3-perfect, and 30240 and 32760 are 4-perfect ([11]). Interestingly, of the more than 5,000 known multiply perfect numbers the only odd one that has been found is 1, which is the only 1-perfect number ([11]).

Theorem 6 (The Only 1-Perfect) *The only 1-perfect number is 1.*

Not only are odd 2-perfect numbers incredibly elusive, but odd multiply perfect numbers aside from 1 are conjectured to be nonexistent as well, leading us to Conjecture 1 which is common among number theorists.

Conjecture 1 *The only odd multiply perfect number is 1.*

The behavior of multiply perfect numbers, and specifically 3-perfect numbers, has direct implications on the odd perfect number problem. Theorem 7

is well-known, easy to prove, and given in [4].

Theorem 7 *An odd number N is 2-perfect if and only if $2N$ is 3-perfect.*

Not too much effort is required to generalize Theorem 7 into Theorem 8.

Theorem 8 *Given an n-perfect number M, if $n \nmid M$ then nM is a $\sigma(n)$-perfect number.*

Unlike Theorem 7, there are some known examples of Theorem 8 in action. For example, 459818240 and 51001180160 are two examples of 3-perfect numbers that are not divisible by 3, meaning that $3 \cdot 459818240 = 1379454720$ and $3 \cdot 51001180160 = 153003540480$ are both 4-perfect numbers since $\sigma(3) = 3 + 1 = 4$.

Interestingly, [16] gives somewhat of a reverse perspective of Theorem 7 as Theorem 9 where if we find a deficient number that satisfies a certain condition we can multiply it with 5 to get an odd perfect number.

Theorem 9 *If $I(n) = \frac{5}{3}$ for some $n \in \mathbb{N}$, then $5n$ is an odd 2-perfect number.*

Note, as Judy A. Holdener does in [16] that Theorem 9 does not give a restriction for the existence of odd perfect numbers but rather provides a way to generate them. A related restriction for the existence of an odd perfect number that was proven by Holdener in [16] is given as Theorem 10.

Theorem 10 *There exists an odd perfect number if and only if there exists $p, n, \alpha \in \mathbb{N}$ such that $p \equiv \alpha \equiv 1 \mod 4$ where p is a prime, $p \nmid n$, and*

$$I(n) = \frac{2p^{\alpha}(p-1)}{p^{\alpha+1}-1}.$$

Holdener's theorem stands out among the many other proofs surrounding odd perfect numbers because, similar to Theorem 7, it successfully makes the existence of an odd perfect number dependent upon the existence of another number with special properties.

Another interesting note on Theorem 7 is the apparent finiteness of n-perfect numbers for each $n > 2$. Conjecture 2 is common in the literature behind multiply perfect numbers.

Conjecture 2 *There are infinitely many 2-perfect numbers but there are finitely many n-perfect numbers for each $n > 2$.*

There are six known 3-perfect numbers and these are generally believed to be all of the triperfects that exist. All six of the triperfects have been known since 1643 ([4]). A way to find all six of the triperfects is to use guess and check by considering numbers of the form $2^a M$ where M is odd and factoring $\sigma(2^a) = 2^{a+1} - 1$ for $a \leq 14$ as suggested in [4]. The six triperfects are

$$120 = 2^3 \cdot 3 \cdot 5$$

$$672 = 2^5 \cdot 3 \cdot 7$$

$$523776 = 2^9 \cdot 3 \cdot 11 \cdot 31$$

$$459818240 = 2^8 \cdot 5 \cdot 7 \cdot 19 \cdot 37 \cdot 73$$

$$1476304896 = 2^{13} \cdot 3 \cdot 11 \cdot 43 \cdot 127$$

$$51001180160 = 2^{14} \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 151.$$

The existence of an odd 2-perfect number suggests that a seventh triperfect number T exists such that $2 \parallel T$. The smallest power of 2 that divides any of the known six triperfect numbers is $2^3 = 8$.

Forms of Perfect Numbers

A common approach to the odd perfect number problem has been to write theorems regarding the form of an OPN or some part of it. For example, [15] gives a theorem from French mathematician Jacques Touchard, which we give as Theorem 11.

Theorem 11 *Any odd perfect number must have the form* $12m+1$ *or* $36m+9$ *for some* $m \in \mathbb{N}$.

It has been well-known for a long time that an odd perfect number must be of the form $4m+1$, which follows directly from Theorem 13. Another result on the form of an odd perfect number is given in Theorem 12 which is from [25].

Theorem 12 *There are no odd perfect numbers of the form* a^a *where* $a \in \mathbb{N}$.

I have done some of my own research pertaining to forms of odd multiply perfect numbers which we will discuss in the Original Work section of this thesis.

Euler proved the equivalent of Theorem 13 as given in [9].

Theorem 13 *If an odd perfect number* N *exists then it is of the form* $N = p^a \cdot s^2$ *where* $p \equiv a \equiv 1 \mod 4$, *p is prime, and* $\gcd(p, s) = 1$.

In this representation of N's form, p is commonly referred to as the 'special prime', as it is in [4]. For my research of studying the forms of odd multiply perfect numbers aside from 2-perfects, I generally refer to any prime that is raised to an odd power in the prime decomposition of an odd multiply perfect number as a 'special prime.'

There are a lot of obscure results for odd perfect numbers that have to do with placing bounds on the largest, smallest, second largest, etc. primes in the prime decomposition of an odd perfect number. For example, Theorem 4 in [4] is given as Theorem 14.

Theorem 14 *If* p *is the special prime in an odd perfect number, then* $\sigma(p+1) \leq 3(p-1)$.

Further, the main theorem in [12], which we give as Theorem 15, pertains to the exponents of an odd perfect number and its smallest prime.

Theorem 15 *Let N be an odd perfect number such that if $p^a \parallel N$ and p is not the special prime then either $3 \mid (a+1)$ or $5 \mid (a+1)$. The smallest prime factor of N belongs to the range $10^8 < p < 10^{1000}$.*

While they are worth noting, I doubt that theorems such as Theorems 14 and 15 will have much weight in the final solution as to whether or not an odd perfect number exists. As William Dunham reflects in [9] on the result that "the third largest prime factor of an odd perfect number of an odd perfect number must exceed one hundred," this result "is especially bizarre, for it tells us something specific about the third largest factor of what could well be a nonexistent entity. This is a bit like knowing the tooth fairy's cousin's middle name."

While the form of odd perfect numbers is often described using modularities (which is defined in Definition 10), the form for even perfect numbers is comparably tight and well-established. Euclid proved that if $2^p - 1$ is prime then $2^{p-1}(2^p - 1)$ is an even perfect number, Euler proved that all even perfect numbers are of Euclid's form, and Italian mathematician Cataldi observed that if p is not prime then $2^p - 1$ is not prime giving us Theorem 16 from [7].

Theorem 16 *All even perfect numbers are equal to $2^{p-1}(2^p - 1)$ for some prime p where $2^p - 1$ is prime.*

A prime of the form $2^p - 1$ is named a Mersenne prime after Marin Mersenne, who, alongside other famous mathematicians such as Fermat, St. Croix, Frenicle, and Descartes, studied multiply perfect numbers in the mid-17th century ([7]). By Theorem 16 there is exactly one even perfect number for every Mersenne prime and vice versa.

Comparing the form of an odd 2-perfect number given in Theorem 13

with that of an even 2-perfect number given in Theorem 16 highlights just how well understood even perfects are vs. odd perfects. For even perfects we have an equation and we know exactly when that equation holds. For odd perfects we have a form that is loosely defined by the modularities of a special prime and its prime power.

Mersenne Primes

There are currently 52 known even perfect numbers. The largest one was discovered in October 2024 by Luke Durant with the Great Internet Mersenne Prime Search (GIMPS) and has upwards of 82,000,000 digits as stated in [19]. While GIMPS is more focused on finding Mersenne primes than perfect numbers, the one-to-one relationship between them means that every new Mersenne prime yields a new even perfect number. The 18 largest and most recently discovered perfect numbers have been discovered by GIMPS since 1996.

Aside from their close association with perfect numbers, Mersenne primes are special in other ways. Currently, eight of the ten largest known primes are Mersenne primes as listed in [24]. As Ray Candlish summarizes in [2], it has been well known for a long time that every even perfect number is a triangular number (defined in Definition 11), which we give as Theorem 17.

Theorem 17 *Every even perfect number is a triangular number.*

Candlish also gives the formula for Theorem 18 in [2].

Theorem 18 *Given* $n \in \mathbb{N}$, $1 + 2 + 3 + ... + n = \frac{1}{2}n(n+1)$*, so every triangular number is of this form.*

Theorem 17 is incredibly easy to prove as, by observation of Theorem 16, every even perfect number is equal to $\frac{1}{2}2^p(2^p - 1)$ for some prime p meaning

that $n = 2^p - 1$ in the form for triangular numbers in Theorem 18. While the overlap between triangular numbers and even perfect numbers is interesting, triangular perfect numbers don't seem to have any significant connections or implications for odd perfect numbers or multiply perfect numbers.

Super Perfect Numbers

In addition to primitive abundant numbers, multiply perfect numbers, spoof perfect numbers, and triangular numbers, yet another group of numbers that shows up in our discussion is the super perfects. We define super perfects in Definition 14.

Definition 14 (Super perfects) *A positive integer n is a super perfect number if and only if* $\sigma(\sigma(n)) = 2n$.

According to [5], "the even super perfects have been completely classified, but it is not known if any odd super perfects exist," showing that, once again, even examples of these numbers are easy to find, but odd examples remain an enigma. We get Theorem 19 from [28].

Theorem 19 *An even super perfect number n is equal to* 2^{p-1} *where* $2^p - 1$ *is a Mersenne prime.*

Similarly to even perfect numbers, even super perfect numbers have a one-to-one relationship with Mersenne primes, which means that there are 52 known super perfects.

We define a further generalization of super perfects in Definition 15, the (m, k)-perfect numbers.

Definition 15 ((m, k)-perfect numbers) *A positive integer n is an* (m, k)*-perfect number if and only if* $\sigma^m(n) = kn$.

Wolfram's page on (m, k)-perfects tells us that there are no even $(m, 2)$-perfect numbers for $m \geq 3$ in [28]. Wolfram also cites J. McCranie on his computational result that there are no $(m, 2)$-perfect numbers less than $4.29 \cdot 10^9$ for any $m \geq 3$ in [28].

A Note on Notation and Conjectures

The author of a work in mathematics is able to define notation however they wish, however it is good practice to stick to common and universally-recognized notation when possible. For the purposes of this thesis I have denoted the sum of divisors function as σ and the abundancy index as I, which is how these two functions are most often referred to in mathematical literature. Another reasonably common way to denote both of these functions is with the sum of positive divisors function defined in Definition 16.

Definition 16 *Define the function σ_z for $z \in \mathbb{C}$ so that*

$$\sigma_z(n) = \sum_{d|n} d^z$$

for $n \in \mathbb{N}$.

Note that, given $n \in \mathbb{N}$, $\sigma_0(n)$ gives the number of divisors of n, $\sigma_1 = \sigma$, and $\sigma_{-1} = I$. While the number of divisors function isn't as relevant to our investigation of perfect numbers and related topics, it is an important function in number theory and it is cool to see how all three of these functions are related. Samuel J. Dittmer makes use of this notation in [8].

In addition to notation, another area of mathematics that is allowed to be subjective is conjectures. 'Conjecture' is just a fancy word for 'guess.' It is best practice for mathematicians to only form conjectures that they believe to be true, but even the best guided conjectures have the potential to be false until proven true. As an example of an untrue conjecture, the last digit

of the even perfects was at one time believed to alternate between 6 and 8 ([7]). This conjecture was based on observation of the first four even perfect numbers, 6, 28, 496, and 8128, which have been recognized as perfect since at least 100 A.D. ([7]). The fifth and sixth even perfects, 33550336 and 8589869056, break this pattern, proving the conjecture false. However, it is true and proven that every even perfect number ends in either a 6 or an 8. We keep the potential wrongness of any statement that has not been rigorously proven in mind as we proceed.

Notes from Dr. Vaaler

Dr. Jeff Vaaler, emeritus professor of mathematics who I met at the ONTD 2023 conference (see appendix), drew my attention to Theorem 323 in [13], which is given here as Theorem 20.

Theorem 20 *Where $e = 2.71828...$ is Euler's number and $\gamma = 0.57721566...$ is the Euler-Mascheroni constant, we have that*

$$\limsup \frac{\sigma(n)}{n \log\log n} = e^{\gamma}.$$

Full understanding of Theorem 20 and lim sup (though we defined lim sup more formally in Definition 12) is out of the scope of this thesis paper, but it essentially means that the constant e^{γ} gives an upper bound for the expression $\frac{\sigma(n)}{n \log\log n}$ and e^{γ} is smaller than any other upper bound that can be found for this expression. Theorem 20 is intriguing and somewhat surprising as it bounds σ, a function which on first impressions seems unpredictable and able to be arbitrarily large.

Another theorem in [13] that caught my attention is Theorem 324, which we give as Theorem 21.

Theorem 21 *The average order of $\sigma(n)$ is $\frac{1}{6}\pi^2 n$.*

Theorem 21 means that, as n approaches infinity, $\sigma(n)$ is on average equal to $\frac{1}{6}\pi^2 n$. Of particular relevance is Corollary 1, which follows from Theorem 21.

Corollary 1 *The average value of $I(n)$ is $\frac{\pi^2}{6}$. That is*

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} \frac{\sigma(n)}{n} = \lim_{X\to\infty} \sum_{x=1}^{X} \frac{1}{x^2} = \zeta(2) = \frac{\pi^2}{6} = 1.644934067...$$

Corollary 1 fascinates me. The abundancy index I seems hard to predict, yet we know what its average is across all the positive integers by Corollary 1. Even stranger, this average value is defined in terms of π, mathematics' most famous irrational number. Further, $\forall n \in \mathbb{N}$, $I(n) \in \mathbb{Q}$ so why is the average of I an element of $\mathbb{Q}'$?

Conclusion

When studying perfect numbers, multiply perfect numbers, primitive abundant numbers, and super perfect numbers there is a common motif. Even examples of these numbers are easy to find a formula for or at least are plentiful while odd ones seems sporadic or non-existent. The frustrating thing about odd perfect numbers, multiply perfect numbers, and super perfect numbers is that we have as of yet been unable to find any odd examples yet proving that odd solutions to these problems do not exist seems to be an almost insurmountable problem. Sylvester compared the odd perfect number problem to the "quadrature of the circle" also known as squaring the circle ([7]). Squaring the circle is a geometry problem that originally interested the ancient Greeks, but has since been proven to be impossible to solve and has become an analogy for an impossible problem. Regardless of whether or the the odd perfect number problem is possible to solve, we proceed to the Original Work section.

Original Work

My original work on the odd perfect number problem can largely be split into three research categories: finding non-divisors of odd multiply perfect numbers, completing computations related to my study using Python code that I wrote, and studying the forms of odd multiply perfect numbers. My work has manifested itself as 6 presentations, 4 songs written for and performed at my presentations, and 2 papers submitted for publication. Summaries and descriptions of my presentations and papers can be found in the Appendices alongside the lyrics to my songs. In this section I will be discussing the theorems, proofs, and conjectures that have emerged from my original work.

Generalization of Sylvester's Proof that an Odd Perfect Number is not Divisible by 105

My research with odd perfect numbers began by looking at Sylvester's Proof that an odd perfect number is not divisible by 105 as given in Theorem 22 from [9].

Theorem 22 *An odd perfect number is not divisible by 105.*

I first encountered Theorem 22 in Winter term 2022 as part of a research project for HNR 276, Honors Mathematics. The number 105 seemed oddly specific to me and I asked the question, are there any other positive integers that can be proven not to divide an odd perfect number?

It turns out that there are! To start, Theorem 23 is well-known, easy to prove, and given in [29].

Theorem 23 *Every multiple of an abundant number is abundant.*

Recall that abundant numbers are defined in Definition 6. From Theorem 23 follows Corollary 2.

Corollary 2 *A perfect number is not divisible by any abundant number.*

So abundant numbers are the trivial OPN non-divisors. However, 105 is not abundant. The abundancy of $105 = 3 \cdot 5 \cdot 7$ is $I(105) = \frac{\sigma(3)\cdot\sigma(5)\cdot\sigma(7)}{3\cdot5\cdot7} = \frac{4\cdot6\cdot8}{3\cdot5\cdot7} = \frac{64}{35} \approx 1.82857 < 2$. In my research I sought to find other non-abundant non-divisors of odd perfect numbers.

I completed an independent study during Spring 2022 to continue my research. As part of the independent study I wrote [29]. At the end of the independent study, I presented at Western Oregon University's Academic Excellence Showcase and submitted my paper for consideration by the Pi Mu Epsilon Journal. At the time that I presented on my research I had discovered 17 non-abundant OPN non-divisors aside from 105. By the time I submitted my paper, I had found a total of 109 non-divisors aside from 105. My paper was fittingly called, "Generalization of Sylvester's Proof that an Odd Perfect Number is not Divisible by 105." In [29], I proved that an odd perfect number cannot be divisible by 2145 (the smallest non-divisor aside from 105) as an example of my generalization. Here I will prove Theorem 24 as an example because 111,111 is my favorite non-divisor that I found.

Theorem 24 *An odd perfect number is not divisible by 111,111.*

Proof: We proceed by way of contradiction. Assume that N is an odd perfect number and $111111 \mid N$. Then, the prime factorization of N will be of the form $N = 3^{k_1} \cdot 7^{k_2} \cdot 11^{k_3} \cdot 13^{k_4} \cdot 37^{k_5} \cdot p_6^{k_6} \cdot ... \cdot p_n^{k_n}$ where every base is a distinct

prime and every k_i is greater than 0. Observe, by Lemma 2,

$$\begin{aligned} I(N) &= \frac{\sigma(N)}{N} \\ &= \frac{\sigma(3^{k_1})}{3^{k_1}} \cdot \frac{\sigma(7^{k_2})}{7^{k_2}} \cdot \frac{11^{k_3}}{11^{k_3}} \cdot \frac{\sigma(13^{k_4})}{13^{k_4}} \cdot \frac{\sigma(37^{k_5})}{37^{k_5}} \cdot \frac{\sigma({p_6}^{k_6})}{{p_6}^{k_6}} \cdots \frac{\sigma({p_n}^{k_n})}{{p_n}^{k_n}} \\ &= \left(1 + \frac{1}{3} + \ldots + \frac{1}{3^{k_1}}\right) \cdot \left(1 + \frac{1}{7} + \ldots + \frac{1}{7^{k_2}}\right) \cdot \left(1 + \frac{1}{11} + \ldots + \frac{1}{11^{k_3}}\right) \\ &\quad \cdot \left(1 + \frac{1}{13} + \ldots + \frac{1}{13^{k_4}}\right) \cdot \left(1 + \frac{1}{37} + \ldots + \frac{1}{37^{k_5}}\right) \\ &\quad \cdot \left(1 + \frac{1}{p_6} + \ldots + \frac{1}{{p_6}^{k_6}}\right) \ldots \left(1 + \frac{1}{p_n} + \ldots + \frac{1}{{p_n}^{k_n}}\right). \end{aligned}$$

Since N is an perfect number, $\sigma(N) = 2N$, and since N is odd, $\sigma(N)$ is divisible by 2 but not 4. Note that, if k_1, k_2, or k_3 is equal to one, then

$$\left(1 + \frac{1}{3} + \ldots + \frac{1}{3^{k_1}}\right) = \frac{4}{3}$$
$$\left(1 + \frac{1}{7} + \ldots + \frac{1}{7^{k_2}}\right) = \frac{8}{7}$$
$$\left(1 + \frac{1}{11} + \ldots + \frac{1}{11^{k_3}}\right) = \frac{12}{11},$$

which would imply that $\sigma(N)$ is divisible by 4. Thus, k_1, k_2, and k_3 are at least 2.

Observe,

$$\begin{aligned} 2 = \frac{\sigma(N)}{N} &\geq \left(1 + \frac{1}{3} + \frac{1}{3^2}\right) \cdot \left(1 + \frac{1}{7} + \frac{1}{7^2}\right) \cdot \left(1 + \frac{1}{11} + \frac{1}{11^2}\right) \cdot \left(1 + \frac{1}{13}\right) \cdot \left(1 + \frac{1}{37}\right) \\ &= \frac{13}{9} \cdot \frac{57}{49} \cdot \frac{133}{121} \cdot \frac{14}{13} \cdot \frac{38}{37} = \frac{52430196}{25666641} > 2, \end{aligned}$$

which is a contradiction. Thus, an odd perfect number is not divisible by 111,111. QED

Note that the proof for Theorem 24 given here follows the structure of Sylvester's proof of Theorem 22 and the proof that I gave to show that an odd perfect number cannot be divisible by 2145 in [29]. The 109 non-divisors that

I found are given as follows from [29].

$3 \cdot 5 \cdot 11 \cdot p$, where p is prime and $13 \leq p \leq 19$

$3 \cdot 5 \cdot 11 \cdot 23 \cdot 29 = 110055$

$3 \cdot 5 \cdot 13 \cdot 17 \cdot 19 = 62985$

$3 \cdot 7 \cdot 11 \cdot 13 \cdot p$, where p is prime and $17 \leq p \leq 179$

$3 \cdot 7 \cdot 11 \cdot 19 \cdot 23 = 100947$

$3 \cdot 7 \cdot 11 \cdot 23 \cdot 31 \cdot 43 = 7082229$

$3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 = 92424189$

$3 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot p$, where p is prime and $41 \leq p \leq 389$

$3 \cdot 13 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 = 46974009365049$

$5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 = 33426748355$

I found these non-divisors with the help of a spreadsheet.

I end [29] with a discussion of primitive abundant numbers, which I defined slightly differently than in Definition 7. I defined primitive abundant numbers as either perfect or abundant numbers that have no perfect or abundant proper divisors. In [29], I end with Conjecture 3.

Conjecture 3 *There are infinitely many distinct numbers that can be proven not to divide an odd perfect number using a generalization of Sylvester's proof.*

A Note on Erdős

Paul Erdős was an amazing mathematician and a prolific writer of mathematical published work. During his life he published around 1,500 papers (a number which remains unsurpassed by any other mathematician) and numerous books. While writing [29] I was trying to determine if I could prove

Conjecture 3 and I realized that I could do so if it had already been proven that there are infinitely many odd primitive abundant numbers. I ran across a book on number theory coauthored by Erdős cited as [26], which, on p.244 listed as an exercise to the reader, "Prove that there are infinitely many odd primitive abundant numbers." This gave me a moment of hope that it had been proven as textbook writers don't usually leave unsolved problems as exercises for the reader. Alas, whether or not there are infinitely many odd primitive abundant numbers remains an open question. Erdős is just the kind of person who would leave such a problem to an unsuspecting undergraduate who is just starting out in number theory.

Computational Response to Vaaler

Dr. Vaaler defined the function $\tilde{\rho} : \mathbb{N} \to \mathbb{Q}$ in [27], which, for our purposes, we define as in Definition 17.

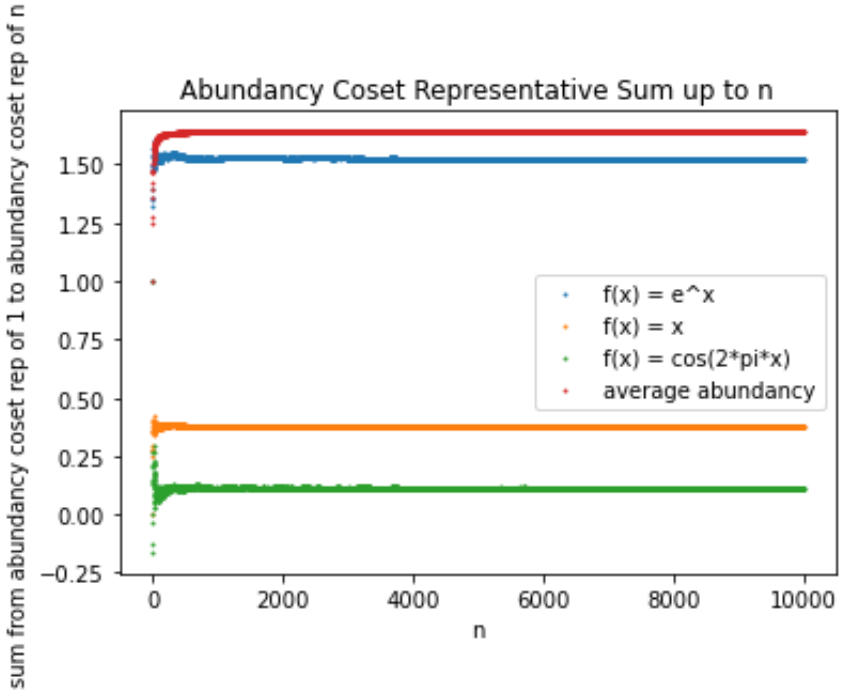
Definition 17 *Define $\tilde{\rho}$ so that, for $n \in \mathbb{N}$, $\tilde{\rho}(n) = I(n) - \lfloor I(n) \rfloor$.*

In [27], Vaaler posed the question, "does the limit

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} \cos 2\pi\tilde{\rho}(n) \tag{4}$$

exist? And if [the limit shown above] does exist, what is the value of the limit?"

I wrote some Python code to compute equation 4 along with some other limits that came to my attention. My code generated the following graph.



This graph shows

$$g_1(N) = N^{-1} \sum_{n=1}^{N} \cos 2\pi\tilde{\rho}(n)$$

$$g_2(N) = N^{-1} \sum_{n=1}^{N} \tilde{\rho}(n)$$

$$g_3(N) = N^{-1} \sum_{n=1}^{N} e^{\tilde{\rho}(n)}$$

$$g_4(N) = N^{-1} \sum_{n=1}^{N} \frac{\sigma(n)}{n}.$$

Observation of this graph and the table of values generated by my code suggests that the limit as $N \to \infty$ exists for each of these functions and leads me

to propose the following estimates.

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} \cos 2\pi\tilde{\rho}(n) \approx 0.113 \tag{5}$$

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} \tilde{\rho}(n) \approx 0.376 \tag{6}$$

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} e^{\tilde{\rho}(n)} \approx 1.52 \tag{7}$$

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} \frac{\sigma(n)}{n} \approx 1.64 \tag{8}$$

From Corollary 1 we have that

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} \frac{\sigma(n)}{n} = \frac{\pi^2}{6} = 1.644934067...$$

which is accurate up to three decimal places with

$$10000^{-1} \sum_{n=1}^{10000} \frac{\sigma(n)}{n} \approx 1.6444958900708022$$

and supports the estimation in equation 8.

Forms of Odd Multiply Perfect Numbers

During the Summer of 2023 I began looking at possible forms that odd n-perfect numbers for values of n aside from 2 could have, and I started to notice some interesting patterns. I wrote my findings on this topic as a paper which has been submitted for publication and which I cite as [14].

In [14], I prove Lemma 5 and use it to define the function υ as in Definition 18.

Lemma 5 *For every $n \in \mathbb{N}$, there exists a unique $k \in \mathbb{Z}_{\geq 0}$ such that $n \equiv 2^k - 1 \mod 2^{k+1}$.*

Definition 18 *Let the function $\upsilon : \mathbb{N} \to \mathbb{Z}_{\geq 0}$ be defined as $\upsilon(n) = j$ where j is the unique nonnegative integer such that $n \equiv 2^j - 1 \mod 2^{j+1}$.*

The theorem that I am most proud of in [14] is Lemma 6. Proving Lemma 6 was the first time that I had the opportunity to use an inductive proof outside of a classroom setting.

Lemma 6 *Let $p \in \mathbb{N}$ be odd and let $j \in \mathbb{N}$. Then, $\forall n \equiv 2^j - 1 \mod 2^{j+1}$,*

$$p^n + ... + p + 1 \equiv 2^{\upsilon(p)+j-1} \mod 2^{\upsilon(p)+j}.$$

Importantly, Corollary 3 follows from Lemma 6.

Corollary 3 *Let p be an odd prime and let $n \in \mathbb{N}$ be odd. Then $\sigma(p^n) \equiv 2^{\upsilon(p)+\upsilon(n)-1} \mod 2^{\upsilon(p)+\upsilon(n)}$.*

Using Corollary 3 and other related corollaries and lemmas, I was able to prove Theorem 25 and Theorem 26, which are given in [14].

Theorem 25 *Given an odd $4k$-perfect number M where k is odd, M is of one of the following three forms where each p_{ij} is a distinct odd prime that does not divide s, $p_{ij} \equiv a_{ij} \equiv 2^i - 1 \mod 2^{i+1}$, and s is odd.*

$$M = {p_{11}}^{a_{11}} \cdot {p_{12}}^{a_{12}} \cdot s^2 \tag{9}$$

$$M = {p_{11}}^{a_{21}} \cdot s^2 \tag{10}$$

$$M = {p_{21}}^{a_{11}} \cdot s^2 \tag{11}$$

Theorem 26 *Given an odd $8k$-perfect number M where k is odd, M is of one of the following six forms where each p_{ij} is a distinct odd prime that does not*

divide s, $p_{ij} \equiv a_{ij} \equiv 2^i - 1 \mod 2^{i+1}$*, and s is odd.*

$$M = {p_{11}}^{a_{11}} \cdot {p_{12}}^{a_{12}} \cdot {p_{13}}^{a_{13}} \cdot s^2 \tag{12}$$

$$M = {p_{11}}^{a_{11}} \cdot {p_{21}}^{a_{12}} \cdot s^2 \tag{13}$$

$$M = {p_{11}}^{a_{11}} \cdot {p_{12}}^{a_{21}} \cdot s^2 \tag{14}$$

$$M = {p_{11}}^{a_{31}} \cdot s^2 \tag{15}$$

$$M = {p_{21}}^{a_{21}} \cdot s^2 \tag{16}$$

$$M = {p_{31}}^{a_{11}} \cdot s^2 \tag{17}$$

I also gave a theorem in [14] which lists all 13 possible forms of a $16k$-perfect number with odd k.

The other part of [14] that I am very proud of is the Final Remarks section. In this section I begin by defining the function ρ as in Definition 19.

Definition 19 *Let* ρ *be defined as, for* $t \in \mathbb{Z}_{\geq 0}$*,*

$$\rho(t) = \left\{ \left\{ (n_1, k_1), (n_2, k_2), ..., (n_j, k_j) \right\} \mid n_i, k_i \in \mathbb{N}, \textit{each } n_i \textit{ is distinct}, \sum_{i=1}^{j} n_i k_i = t \right\}.$$

Given $t \in \mathbb{Z}_{\geq 0}$, $\rho(t)$ is my way of expressing the set of integer partitions of t. An integer partition is a collection of positive integers that can be summed together to equal a given integer. For example, the five distinct integer partitions of 4 are 4, 3+1, 2+2, 2+1+1, and 1+1+1+1, the seven distinct integer partitions of 5 are 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, and 1+1+1+1+1, etc. My way of expressing an integer partition is as a set of tuples (n, k) where each n is a positive integer that appears in the partition and k is the number of times it appears, where each n is distinct. For example, the five distinct elements of $\rho(4)$ aka the integer partitions of 4, given in the same order as before, are $\{(4, 1)\}$, $\{(3, 1), (1, 1)\}$, $\{(2, 2)\}$, $\{(2, 1), (1, 2)\}$, and $\{(1, 4)\}$, the seven distinct elements of $\rho(5)$ aka the integer partitions of 5 are $\{(5, 1)\}$,

$\{(4,1),(1,1)\}$, $\{(3,1),(2,1)\}$, $\{(3,1),(1,2)\}$, $\{(2,2),(1,1)\}$, $\{(2,1),(1,3)\}$, and $\{(1,5)\}$, etc. Note that $\rho(0)$ equals the empty set.

I then define the function ω as in Definition 20.

Definition 20 *Define $\omega : \mathbb{Z}_{\geq 0} \to \mathbb{N}$ so that, for $t \in \mathbb{Z}_{\geq 0}$*

$$\omega(t) = \sum_{A \in \rho(t)} \prod_{(n,k) \in A} \binom{n+k-1}{k}$$

Using the ω-function defined as in Definition 20, I proved Theorem 27 in [14].

Theorem 27 *A $2^t k$-perfect number M with odd k and $t \in \mathbb{Z}_{\geq 0}$ has $\omega(t)$ possible forms.*

I calculated $\omega(n)$ from $n = 0$ to 11 and I got the following sequence of values, as given in [14].

$$1, 1, 3, 6, 13, 24, 48, 86, 160, 282, 500, 859, ...$$

Interestingly, this sequence aligns with OEIS A000219 ([21]). The On-line Encyclopedia of Integer Sequences is a searchable catalog of sequences that have been of interest to a mathematician at one time or another. It is valuable because, when a mathematician comes across a sequence, they can easily and quickly check if the sequence has a known equation, has been studied, or has appeard in any literature by searching for it within OEIS.

OEIS A000219 is the number of planar partitions of n. A planar partition of n is a two dimensional partition of n, where the rows and columns decrease from left to right and top to bottom. For example,

3	2	1	.	3	3	.	1	1	1	.	6	3	.	9	.	4
1	1		.	2		.	1	1	1	.			.		.	4
1			.	1		.	1	1	1	.			.		.	1

represent six valid planar partitions of 9 since

$$\begin{aligned} 9 &= (3+1+1)+(2+1)+(1) \\ &= (3+2+1)+3 \\ &= (1+1+1)+(1+1+1)+(1+1+1) \\ &= (6)+(3) \\ &= (9) \\ &= (4+4+1) \end{aligned}$$

whereas three invalid examples of planar partitions are

$$\begin{array}{ccccccccccc} 1 & 2 & 1 & . & 3 & 3 & . & 1 & 1 & 1 \\ 3 & 1 & & . & 2 & & . & 1 & 1 & 1 \\ 1 & & & . & & 1 & . & 1 & 2 & \end{array}$$

because not all rows and columns decrease from left to right and top to bottom.

Noticing that my sequence aligned with OEIS A000219 led me to Conjecture 4 ([14]).

Conjecture 4 *The number of planar partitions of $n \in \mathbb{Z}_{\geq 0}$ is equal to $\omega(n)$.*

I tried to prove Conjecture 4 by finding a surjective mapping from the set of planar partitions of $n \in \mathbb{Z}_{\geq 0}$ to each possible form of a $2^n k$-perfect number with odd k, but was not successful. I have come to the conclusion that proving Conjecture 4 is out of the scope of my research.

Songs and Presentations

I completed six presentations on OPNs and MPNs. A summary and abstract of each presentation is given in the appendix. I wrote a total of four songs on perfect numbers which I performed during my presentations. I found that

breaking into song in the middle or at the end of my talks was a good way to engage my audience, make the subject matter more memorable, and make my talks more exciting. My audiences responded well to the songs and my musical interludes were always met by applause. The lyrics to each of my songs can also be found in the appendix.

Final Remarks

For my first paper, cited as [29] and published in Fall 2022, I won the Richard V. Andree Award, which is "given annually to the authors of the papers, written by undergraduate students, that have been judged by the officers and councilors of Pi Mu Epsilon to be the best that have appeared in the Pi Mu Epsilon Journal in the past year" ([22]).

My second paper has been submitted for publication and is awaiting a decision from the editor. I had intended to write a third paper on possible non-divisors of odd triperfect numbers, but found that that had already been covered by [17]. I also considered creating a database of known multiply perfect numbers but found that that had been done by [11].

The odd perfect number problem remains to be solved. New advancements on the problem are made regularly and I am hopeful that it will be solved in my lifetime. I also hope that the contributions that I have made to the problem will be a part of the solution for the OPN problem, but I know that this is unlikely. There are many approaches that number theorists have taken to the OPN problem and I have only been able to study a few.

In addition to OPNs, we looked at multiply perfect numbers, the sigma function and the abundancy index, primitive abundant numbers, super perfects, and spoof perfects. For many of these numbers, studying even examples is easier than studying odd examples and odd instances tend to have a lot more open questions surrounding them. Similar to the OPN problem, it will be interesting to see what advancements in our understanding of the other abundancy-defined numbers will emerge in coming years. I expect that studying these other numbers is how the OPN problem will be solved.

References

[1] H. A. Bernhard. On the least possible odd perfect number. *The American Mathematical Monthly*, 59(9):628–629, November 1949.

[2] Ray Candlish. Mersenne primes, perfect numbers and triangular numbers. *Mathematics in School*, 35(4):32–33, September 2006.

[3] W. Edwin Clark. *Elementary Number Theory*. University of South Florida, 2003.

[4] Graeme L. Cohen and Ronald M. Sorli. On odd perfect numbers and even 3-perfect numbers. *Integers*, 2011.

[5] G. G. Dandapat, J. L. Hunsucker, and Carl Pomerance. Some new results on odd perfect numbers. *Pacific Journal of Mathematics*, 57(2), 1975.

[6] Leonard Eugene Dickson. Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors. *American Journal of Mathematics*, 35(4):413–422, October 1913.

[7] Leonard Eugene Dickson. *History of the Theory of Numbers*, volume 1. AMS Chelsea Publishing, Providence, Rhode Island, 1992.

[8] Samuel J. Dittmer. Spoof odd perfect numbers. *Mathematics of Computation*, 83(289):2575–2582, September 2014.

[9] William Dunham. Odd perfect numbers: A triptych. *The Mathematical Intelligencer*, 42:42–46, 2020.

[10] Luke Durant. Gimps discovers largest known prime number: $2^{136279841} - 1$. https://www.mersenne.org/primes/?press=M136279841, October 2024.

[11] Achim Flammenkamp. The multiply perfect numbers page. Accessed online February 23, 2025 at https://wwwhomes.uni-bielefeld.de/achim/mpn.html, 2023.

[12] S. Adam Fletcher, Pace P. Nielson, and Pascal Ochem. Sieve methods for odd perfect numbers. *Mathematics of Computation*, 81(279):1753–1776, July 2012.

[13] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. Clarendon Press, Oxford, 1968.

[14] Jasmine Wetter Hiebert. Forms of multiply perfect numbers. *Not yet published*, 2025.

[15] Judy A. Holdener. A theorem of touchard on the forms of odd perfect numbers. *The American Mathematical Monthly*, (7):661–663, 2002.

[16] Judy A. Holdener. Conditions equivalent to the existence of odd perfect numbers. *Mathematics Magazine*, 79(5):389–391, December 2006.

[17] Masao Kishore. Odd triperfect numbers. *East Carolina University Department of Mathematics*, 1984.

[18] Steve Nadis. Mathematicians opan a new front on an ancient number problem. Accessed online February 2025 at https://www.quantamagazine.org/mathematicians-open-a-new-front-on-an-ancient-number-problem-20200910/: :text=Mathematicians2020.

[19] Numberphile. The man who found the world's biggest prime. Accessed online February 2025 at https://www.youtube.com/watch?v=Yp4ilFOtoeg, 2024.

[20] Pascal Ochem and Michaél Rao. Odd perfect numbers are greater than 10^{1500}. *Mathematics of Computation*, 81(279):1869–1877, July 2012.

[21] OEIS Foundation Inc. Number of plane partitions (or planar partitions) of n. Entry A000219 in The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A000219, 2025.

[22] Pi Mu Epsilon. Richard v. andree awards. Accessed online at https://pme-math.org/richard-v-andree-awards: :text=Theon March 15n 2025, 2025.

[23] Paul Pollack. On dickson's theorem concerning odd perfect numbers. *The American Mathematical Monthly*, 118(2):161–164, February 2011.

[24] PrimePages. The largest known primes–a summary. Accessed online at https://t5k.org/largest.html, February 2025.

[25] T. M. Putnam. Perfect numbers. *The American Mathematical Monthly*, 17(8/9):165–168, 1910.

[26] Janos Suranyi and Paul Erdős. *Topics in the Theory of Numbers*. Springer Science & Business Media, 2003.

[27] Jeff Vaaler. Notes on perfect numbers. Personal communication, February 2023.

[28] Eric W. Weisstein. Superperfect number. Accessed online at https://mathworld.wolfram.com/SuperperfectNumber.html, February 2025.

[29] Jasmine Wetter. Generalization of sylvester's proof that an odd perfect number is not divisible by 105. *Pi Mu Epsilon Journal*, 2022.

Appendix

Songs

Song #1 "The Perfect Number Song"

Originally performed at AES 2022. Note that this song is out of date, as there are now 52 known perfect numbers.

Let me sing you a song about... Perfect Numbers,
Equal to the sum of their... proper factors.

Even ones we know exist
We've found 51 including 6, 28, and 496,
And the largest one has over 49 million digits.

Odd perfect numbers are harder to find
Mathematicians have been searching for a very long time
Can we find an odd perfect number?
Or at least prove their nonexistence forever?

We must learn from our mistakes.
Learn from the wrong paths we may take.
And someday, someone, will solve...
The mystery of the odd perfect number.

Song #2 "Odd Perfect Numbers"

Originally performed as the finale of AES 2022.

Odd perfect numbers, a passion of mine.
Odd perfect numbers, so hard to find.

Now we know a bunch of numbers you're not divisible by
Including, but not limited to 105.
I'm almost certain you don't exist, but I wish I knew why.
Odd perfect numbers, so hard to find.

Odd perfect numbers,
We've searched high and we've searched low,
And now we know that there are none below
Ten to the 2000th power
And that's a big number.
Odd perfect numbers, so hard to find.

Song #3 "Multiply Perfect"

Originally performed at AES 2024

Multiply perfect...
Divisor of the sum of your factors
We want to prove all the conjectures
But you're just a number, just an integer
Just a natural

Everyone wants to find you
Everyone wants your primes
Everyone wants your form

They want to know you, to study you
You are their muse

”I’m not so special!” you declare.
”My abundancy is an integer, but that’s all that’s there.
Why must my identity be based on something that’s just part of me.
Why do I have to justify the reason that I am me.”

One is the only odd one that we know
And one seems to like being alone
The search for odd 2-perfects has been big and in vain.
The search for even 2-perfects seems to leave room for gain.

Triperfects, or 3-perfect if you prefer,
Six are known
But are they alone?
If they are, there are no OPNs.
And that would bring that question to an end.

Multiply perfect...
Divisor of the sum of your factors
So many conjectures
But you’re just a number, just an integer,
Just a natural

Song #4 “Multiply Perfect (ver. 2)”

Originally performed at JMM 2025

Multiply perfect...
Divisor of the sum of your factors
Your abundancy is whole
But you’re just a number, just an integer
Just a natural

Everyone wants to find you
Everyone wants your primes
Everyone wants your form
They want to know you, to study you
You are their muse

One is the only odd one that we know
And one seems to like being alone
The search for odd perfects has been big and in vain
The search for even perfects seems to leave room for gain

Triperfect, or 3-perfect if you prefer,
Six are known but are they alone?
Do they have any other friends?
One day we’ll bring this question to an end

Multiply perfect...
Divisor of the sum of your factors
So many conjectures
But you’re just a number, just an integer
Just a natural

Summary of Presentations

Following is a list of the presentations I have given on odd perfect numbers and related topics in chronological order.

(Western Oregon University Academic Excellence Showcase 2022) "The Mathematical Mystery of Odd Perfect Numbers," May 26, 2022.

The information in this presentation is covered and then some by my first paper, published Fall 2022. The main result is generalizing Sylvester's 105 proof and finding a bunch of other 'odd perfect number non-divisibility numbers,' as I call them.

Abstract. Perfect numbers are numbers that are equal to the sum of their proper factors (for example, 28 is a perfect number because 28 = 1 + 2 + 4 + 7 + 14). We will discuss some of the characteristics of perfect numbers, specifically those that relate to the open question of the existence of odd perfect numbers. This project expands upon previous results regarding the divisibility of odd perfect numbers.

(Oregon Number Theory Days hosted by Oregon State University 2023) "How to find Integers that Cannot Divide Elusive Odd Perfect Numbers: Generalizing Sylvester's Proof," February 18, 2023.

This presentation covered most of the same information as my AES 2022 presentation, except by this point my paper had been finalized and published and I had found even more non-divisibility numbers. At this conference I met Dr. Jeff r, emeritus professor of mathematics at OSU, who I sent a draft copy of my paper to over email. He responded with four pages of notes and a question which he posed to me in response to my paper. I responded to his notes and attempted to answer his question to the best of my ability during summer 2023.

Abstract. Odd perfect numbers have been of interest to mathematicians for millennia. While they are generally believed not to exist, no one has been able to prove their nonexistence. Many discoveries have been made regarding the characteristics of odd perfect numbers, including James Sylvester's 1888 proof that no odd perfect number is divisible by 105. We will discuss a generalization of Sylvester's proof that allows us to prove that 2145 and at least 108 other integers aside from 105 are impossible odd perfect number divisors.

(Northwest Undergraduate Mathematics Symposium 2023, held virtually) "How to find Integers that Cannot Divide Odd Perfect Numbers using a Generalization of Sylvester's Proof," November 11, 2023.

This was a repeat of information from my last two presentations. In all three presentations I sang "The Perfect Numbers Song" and "Odd Perfect Numbers." The songs got wonderful responses in all my presentations. All the mathematicians present loved to hear some number theory set to music, which does not happen very often.

Abstract. A widely known and thought about open question in number theory is whether or not an odd perfect number, that is an odd positive integer equal to the sum of its proper divisors, exists. In 1888, James Joseph Sylvester proved that such a number cannot be divisible by 105. This presentation includes an extension of Sylvester's proof discovered recently by the speaker and lists more integers that can be proven not to divide an odd perfect number. We will also discuss how to find such integers.

(BOT Meeting) "Divisibility Conditions for Odd Perfect Numbers," April 16, 2024.

I was asked to give a five minute presentation before WOU's Board of

Trustees demonstrating the research I have done. At the end I was asked to perform one of my songs. I sang "Odd Perfect Numbers" from memory, without practice. It was fun and well-received.

Abstract. No abstract.

(Western Oregon University Academic Excellence Showcase 2024) "The Mathematical Mystery of Odd (Tri)Perfect Numbers," May 30, 2024.

In this presentation I dropped new research which I completed for the event. I found an odd triperfect number non-divisibility number.

Abstract. This is a continuation of the AES 2022 presentation "The Mathematical Mystery of Odd Perfect Numbers." The first presentation shared a generalization of J.J. Sylvester's 1888 proof that an odd perfect number, should one exist, cannot be divisible by 105. Sylvester's proof had not been generalized prior to this point. This second presentation will look at extending Sylvester's proof to find integers that cannot divide odd triperfect numbers. This is a new result in number theory.

(Joint Mathematics Meeting 2025) "Odd Perfects, Multiply Perfects, and Non-Divisors, oh my! An Exploration of the Odd Perfect Number Problem," January 9, 2025.

I gave this presentation at by far the largest conference that I have ever been to. In my presentation I discuss non-divisors of odd 2-perfect, 3-perfect, and 4-perfect numbers. I also shared some new conjectures about the forms of odd multiply perfect numbers. I referenced the work of Sylvester and Euler.

Abstract. Perfect numbers, and the broader collection of multiply perfect numbers are a much studied yet little understood facet of number theory. The infamous OPN problem can be simply stated but has proven to be one of the hardest problems in number theory to solve. The question behind it is: do

odd perfect numbers exist? Of the 51 known perfect numbers (the smallest being 6 and the largest being $2^{82589932} \times (2^{82589932} - 1)$)) all are even. There is only one out of the known 5932 multiply perfect numbers that is odd. Most number theorists conjecture that there is one and only one odd multiply perfect number but there are infinitely many even multiply perfect numbers. In 1888, James Joseph Sylvester proved that if an OPN exists, then it cannot be divisible by 105. We extend Sylvester's result to other odd multiply perfect numbers, illuminate other values aside from 105 that can be proven to be an odd perfect number non-divisor, and attempt to shed some new light on this very old problem.

Summary of Papers

"Generalization of Sylvester's Proof that an Odd Perfect Number is not Divisible by 105," Pi Mu Epsilon Journal, Vol. 15, No. 7, (2022).

With this paper I won the 2022 Richard V. Andree Award, which is "given annually to the authors of the papers, written by undergraduate students, that have been judged by the officers and councilors of Pi Mu Epsilon to be the best that have appeared in the Pi Mu Epsilon Journal in the past year" ([22]). I completed this paper under the edit suggestions and research guidance of Dr. Leanne Merrill during an independent study completed in Winter 2022.

Abstract. Odd perfect numbers have been of interest to mathematicians for millennia. Many discoveries have been made regarding the characteristics of odd perfect numbers, including James Sylvester's 1888 proof that no odd perfect number is divisible by 105. This article, which generalizes Sylvester's proof, proves that an odd perfect number is also not divisible by 2145 and discusses the extension of this result to at least 108 other impossible divisors.

"Forms of Odd Multiply Perfect Numbers," unpublished, (2025).

I completed this paper under the guidance of Dr. Cheryl Beaver during an independent study completed in Winter 2025.

Abstract. The form of an odd 2-perfect number was established by Euler in the 18th century to be $N = p^a \cdot s^2$ where p is prime, $p \equiv a \equiv 1 \mod 4$, and $\gcd(p, s) = 1$. We will prove that odd 4-perfect numbers, 8-perfect numbers, and 16-perfect numbers have exactly 3, 6, and 13 possible forms, respectively. We will give these possible forms and provide an equation which we conjecture to count the number of possible forms of other odd MPNs.